

Interpolating coherent states for Heisenberg-Weyl and single-photon $SU(1,1)$ algebras

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Short title

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Abstract

New quantal states which interpolate between the coherent states of the Heisenberg-Weyl (W_3) and $SU(1,1)$ algebras are introduced. The interpolating states are obtained as the coherent states of a closed and symmetric algebra which interpolates between the W_3 and $SU(1,1)$ algebras. The overcompleteness of the interpolating coherent states is established. Differential operator representations in suitable spaces of entire functions are given for the generators of the algebra. A nonsymmetric set of operators to realize the W_3 algebra is provided and the relevant coherent states are studied.

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1 Introduction

An extremely useful mathematical framework for dealing with continuous symmetries is the theory of Lie groups and Lie algebras. In the context of quantum optics, the use of group theory has been very prominent ever since the discovery of the coherent states of electromagnetic field [1, 2, 3, 4]. The usual coherent states are the unitarily displaced vacuum state of the harmonic oscillator. The unitary displacement is effected by the displacement operator $D(\alpha)$ given by $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, \hat{a}^\dagger and \hat{a} being the creation and annihilation operators respectively. Each coherent state is characterized by a complex number α and the state is expressed in terms of the Fock (number) states as

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (1)$$

The algebra relevant to these states is the Heisenberg-Weyl algebra W_3 , generated by the operators \hat{a} , \hat{a}^\dagger and the identity operator I . These three operators form a closed algebra as $[\hat{a}, \hat{a}^\dagger] = I$. (The Heisenberg-Weyl algebra can be extended to W_4 by including the number operator $\hat{a}^\dagger\hat{a}$ and the algebra is still closed.) It turns out that the states $|\alpha\rangle$ are eigenstates of \hat{a} , one of the elements of the algebra. Such eigenstates are algebraic coherent states. States obtained, as in Eq.(1), by an unitary transformation are said to be group-theoretic coherent states or coherent states in the sense of Perelomov. The most notable feature of coherent states is their overcompleteness and it is mathematically expressed as

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = I. \quad (2)$$

The integration is over the entire complex plane and the integration measure $d^2\alpha$ is $d(Re(\alpha))d(Im(\alpha))$.

The two-photon operators \hat{a}^2 and $\hat{a}^{\dagger 2}$ and the number operator $\hat{a}^\dagger\hat{a}$ form a closed algebra which is identical to the well-known $SU(1,1)$ algebra of three operators K_0 , K_+ and K_- which satisfy

$$[K_0, K_\pm] = \pm K_\pm \quad [K_+, K_-] = -2K_0. \quad (3)$$

In the case of two-photon operators, K_0 , K_+ and K_- are identified with $(2\hat{a}^\dagger\hat{a} + 1)/4$, $\hat{a}^{\dagger 2}/2$ and $\hat{a}^2/2$ respectively. The algebraic coherent states for this realization are the even and odd coherent states [5]. The group-theoretic coherent are the squeezed vacuum and first excited states [6, 7, 8]. It is, however, possible to realize $SU(1,1)$ algebra with deformed single-photon operators [9, 10, 11, 12, 13, 14]. Here, deformation implies that the generators are multiplied by an operator-valued function of the number operator. Consider the Holstein-Primakoff realization

$$K_0 = \hat{a}^\dagger\hat{a} + j \quad K_- = \sqrt{\hat{a}^\dagger\hat{a} + 2j}\hat{a} \quad K_+ = \hat{a}^\dagger\sqrt{\hat{a}^\dagger\hat{a} + 2j}. \quad (4)$$

The deforming operator is $\sqrt{\hat{a}^\dagger \hat{a} + 2j}$. The Casimir invariant for this realization is $j(j-1)$. The operator realizations indeed satisfy the $SU(1,1)$ algebra for all values of j , but they are not two-photon operators. Here j is a constant and we set it equal to $\frac{1}{2}$ in the following discussion. The algebraic coherent states, defined as the eigenstates of K_- , for this realization are

$$|\alpha, 1\rangle\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |n\rangle, \quad |z| < 1. \quad (5)$$

The normalization constant N is $1/\sqrt{I_0(2|\alpha|)}$ and I_0 is the Bessel function of second kind of order zero [15]. These coherent states correspond to the "discrete" representation where the parameter j is $1/2$. In general, j can be either integer or half-integer for discrete representation. Eigenstates of deformed annihilation operators are termed "nonlinear coherent states" as they can be thought of as the coherent states of an oscillator with energy-dependent frequency [16, 17, 18].

The group-theoretic coherent states are constructed as

$$\begin{aligned} |\alpha, 1\rangle_p &= \exp(\alpha K_+ - \alpha^* K_-) |0\rangle, \\ &= \frac{1}{\sqrt{1-|\zeta|^2}} \sum_{n=0}^{\infty} \zeta^n |n\rangle. \end{aligned} \quad (6)$$

The parameter ζ is a function of $\alpha = |\alpha| \exp(i\theta)$ and the relation is $\zeta = \exp(i\theta) \tanh(|\alpha|)$.

The two realizations of $SU(1,1)$ algebra, one in terms of two-photon operators and another in terms of deformed single-photon operators, are useful in solving the intensity-dependent Jaynes-Cummings model (JCM) [19]. The two-photon realization of $SU(1,1)$ algebra is relevant if the interaction between a single-mode cavity field and a two-level atom is described by

$$\hat{H}_{int} = g(\sigma_- \hat{a}^{\dagger 2} + \sigma_+ \hat{a}^2). \quad (7)$$

The operators σ_{\pm} are the raising and lowering operators for the two levels of the atom. For Holstein-Primikoff realization (with $j = 0$), the relevant interaction is

$$\hat{H}_{int} = g(\sigma_- \hat{a}^\dagger \sqrt{\hat{a}^\dagger \hat{a}} + \sigma_+ \sqrt{\hat{a}^\dagger \hat{a}} \hat{a}). \quad (8)$$

The interaction describes single-photon processes with nonlinear coupling between the atom and the field. The later case has been extensively studied in the context of JCM which exhibits complete periodicity in population inversion of atomic levels and fields with infinite statistics [20].

A more general case of interaction is

$$\hat{H}_{int} = g(\sigma_- \hat{a}^\dagger \sqrt{k \hat{a}^\dagger \hat{a} + 1} + \sigma_+ \sqrt{k \hat{a}^\dagger \hat{a} + 1} \hat{a}). \quad (9)$$

In the absence of nonlinear coupling, obtained by setting $k = 0$, this Hamiltonian describes the usual JCM. In the following section we consider operators which

are relevant for the Hamiltonian given in Eq.(9). In the next section, it is shown that the operators satisfy the W_3 or $SU(1,1)$ algebra depending on whether k is zero or unity. We construct coherent states for the algebra satisfied by the operators in the above Hamiltonian and study the properties of the states as a function of k . The overcompleteness of the states are proven for both the algebraic and group-theoretic coherent states. Properties such as energy fluctuations, quadrature squeezing are studied. In Section III, we introduce nonsymmetric set of operators to realize W_3 algebra and study the relevant coherent states.

2 Generalization of single-photon $SU(1,1)$ coherent states

We introduce an additional parameter k (nonnegative and less than or equal to unity) in the symmetric set of operators defined in Eq. (4) such that the $SU(1,1)$ realization is obtained when $k = 1$. The symmetric set of operators is

$$A_0 = k\hat{a}^\dagger\hat{a} + \frac{1}{2}, \quad A_- = \sqrt{k\hat{a}^\dagger\hat{a} + 1}\hat{a}, \quad \text{and} \quad A_+ = \hat{a}^\dagger\sqrt{k\hat{a}^\dagger\hat{a} + 1}. \quad (10)$$

These operators are closed under commutation, and we have

$$[A_0, A_\pm] = \pm k A_\pm \quad [A_+, A_-] = -2A_0. \quad (11)$$

The Casimir invariant of this closed algebra is $A_0^2 - (k/2)\{A_-, A_+\} = \frac{1}{2}(\frac{1}{2} - k)$, where $\{A_-, A_+\}$ stands for anticommutation of the two operators. Successive eigenvalues of the Casimir operator differ by k .

Two important limiting cases of the above commutation relations are when k takes the values zero and unity respectively. In the former case, the algebra reduces to the Heisenberg-Weyl algebra W_3 and in the later case it becomes the $SU(1,1)$ algebra. Thus, the algebra can be thought of as interpolating between the $SU(1,1)$ and W_3 algebras. The fact that one algebra can be obtained from another algebra is known as "contraction" and the procedure to go from $SU(1,1)$ to W_3 is known [21, 22, 23, 24, 25]. What we have presented here is a *realization* of an algebra which has W_3 and $SU(1,1)$ as the limiting cases.

2.1 Algebraic coherent states

The algebraic coherent states for the algebra are defined by

$$A_-|\alpha, k\rangle = \alpha|\alpha, k\rangle, \quad (12)$$

and the number state expansion is

$$|\alpha, k\rangle = N_k \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!k^n(\frac{1}{k})_n}} |n\rangle. \quad (13)$$

The states are normalizable for all values of α and the normalization constant N_k is given by

$$N_k^2 = \left(\frac{|\alpha|}{\sqrt{k}} \right)^{1-\frac{1}{k}} \Gamma\left(\frac{1}{k}\right) I_{1-\frac{1}{k}} \left(\frac{2|\alpha|}{\sqrt{k}} \right) \quad (14)$$

The symbol $(\frac{1}{k})_n$ is Pochhammer notation for the product $\prod_{j=1}^n (\frac{j}{k})$ and $(\frac{1}{k})_0 = 1$ [15]. As expected, in the limit of k becoming zero the states $|\alpha, k\rangle$ become the usual coherent states $|\alpha\rangle$.

Now, we prove the completeness relation for the states $|\alpha, k\rangle$ ($0 < k \leq 1$). When $k = 0$ the states are the usual coherent states and they are overcomplete (refer Eq.2). We need to show that

$$\frac{1}{\pi} \int d\mu |\alpha, k\rangle \langle \alpha, k| = I, \quad (15)$$

for some suitable integration measure μ . This problem naturally leads to the *problem of moments* wherein it is required to construct a probability distribution from the knowledge of its moments[26, 27]. Substituting the number state expansion in Eq.(13) into Eq. (15), the lhs of the later equation becomes

$$\text{lhs} = \frac{1}{\pi} \int d\mu N_k^2 \sum_{n,m=0}^{\infty} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!k^{n+m}(\frac{1}{k})_n(\frac{1}{k})_m}} |n\rangle \langle m|. \quad (16)$$

On substituting $\alpha = r \exp(i\theta)$ and setting $d\mu = \frac{\rho(r)r}{N_k^2} dr d\theta$, the completeness relation becomes

$$I = \frac{1}{\pi} \int \rho(r) r dr \sum_{n=0}^{\infty} \frac{r^{2n}}{n! k^n (\frac{1}{k})_n} |n\rangle \langle n|. \quad (17)$$

For the equation to be valid, the condition is

$$\int \rho(x) x^n dx = \frac{2\Gamma(n+1)\Gamma(\frac{1}{k}+n)}{\Gamma(\frac{1}{k})}, \quad (18)$$

with $x = r^2$.

By comparing with the standard formula [28]

$$\int_0^{\infty} x^{s-1} x^{(1/k-1)/2} K_{\frac{1}{k}-1}(2\sqrt{x}/k) dx = \frac{1}{2} k^s \Gamma(s + \frac{1}{k} - 1) \Gamma(s), \quad (19)$$

we infer

$$\rho(r) = \frac{2}{k\Gamma(\frac{1}{k})} r^{(\frac{1}{k}-1)} K_{\frac{1}{2}(\frac{1}{k}-1)}\left(\frac{2}{k}r\right). \quad (20)$$

Here K_ν is the modified Bessel function of order ν [15]. Thus, the states $|\alpha, k\rangle$ provide a resolution of identity. The inner product between two states, say $|\alpha, k\rangle$ and $|\beta, k\rangle$, is

$$\langle \alpha, k | \beta, k \rangle = (\alpha^* \beta)^{\frac{k-1}{2k}} \frac{I_{\frac{1-k}{k}}(2\sqrt{\alpha^* \beta / k})}{\sqrt{|\alpha \beta|^{\frac{k-1}{k}} I_{\frac{1-k}{k}}(2|\beta|/\sqrt{k}) I_{\frac{1-k}{k}}(2|\alpha|/\sqrt{k})}}. \quad (21)$$

The states corresponding to two different values of α are not orthogonal and hence the states $|\alpha, k\rangle$ are overcomplete.

The uniqueness of the weight function $\rho(r)$ is guaranteed if the moments μ_n ($n = 0, 1, 2, \dots$) satisfy the sufficient condition

$$\sum_{n=0}^{\infty} \mu_n^{-\frac{1}{2n}} = \infty. \quad (22)$$

For the present case the moments are given by Eq.(18) and they satisfy the sufficient condition. Therefore, the weight function $\rho(r)$ is unique. Recently, a variety of coherent states have been constructed based on the solution of moments problem using the tabulated inverse Mellin transforms[29].

Any harmonic oscillator state $|\psi\rangle$ can be expanded in terms of the overcomplete set of states $|\alpha, k\rangle$ as

$$|\psi\rangle = \frac{1}{\pi} \int d\mu f(\alpha^*) |\alpha, k\rangle. \quad (23)$$

The function $f(\alpha^*)$ is $\langle \alpha, k | \psi \rangle$, the projection of the state $|\psi\rangle$ on the eigenstates of A_- . In the space of functions $N_k^{-1}(|\alpha|)f(\alpha^*)$, the generators A_- , A_+ and A_0 are represented as

$$A_+ = \alpha^* \quad A_- = \frac{d}{d\alpha^*} + k\alpha^* \frac{d^2}{d\alpha^{*2}} \quad A_0 = k\alpha^* \frac{d}{d\alpha^*} + \frac{1}{2}. \quad (24)$$

The two limiting cases of $|\alpha, k\rangle$ are the coherent states $|\alpha\rangle$ and the states $|\alpha, 1\rangle$ corresponding to k becoming zero and unity respectively. Thus, the states corresponding to other values of k interpolate between the two limiting cases. The term "interpolating states" seems appropriate as the properties of the states are intermediate between those of the limiting cases. For instance, the coherent states are Poissonian, meaning that the variance and mean of the number distribution are equal. The single-photon SU(1,1) coherent states are sub-Poissonian, *i.e.*, the variance is less than the mean for their number distribution. The states corresponding to arbitrary k ($\neq 0$) are also sub-Poissonian. A quantitative measure for the deviation from Poissonian behaviour is the Q parameter[30], defined as

$$Q = \frac{\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle}. \quad (25)$$

In Fig. 1 the variation of Q as a function of k is shown as a function of $|\alpha|$ for different values of k . The states $|\alpha, k\rangle$ are sub-Poissonian for $0 < k \leq 1$.

The algebraic coherent states exhibit squeezing in both the field quadratures, namely,

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \text{and} \quad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}. \quad (26)$$

For the coherent states $|\alpha\rangle$ the uncertainties in x and p are same as those of the vacuum state. In the case of $|\alpha, k\rangle$, the squeezing in the x quadrature increases with both α and k . The dependence is depicted in Figs. 2a-2d where the variation of Δx with α is shown for various values of k . It is interesting to note that the uncertainty profiles are symmetric under $\alpha \rightarrow -\alpha$. This can be understood as follows. The symbol $\langle \dots \rangle$ stands for the expectation value in the states $|\alpha, k\rangle$. In terms of the operators \hat{a} and \hat{a}^\dagger , the uncertainty in x is

$$(\Delta x)^2 = \frac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a}^\dagger \rangle^2 - \langle \hat{a} \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle], \quad (27)$$

and that in p is

$$(\Delta p)^2 = \frac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \rangle^2 + \langle \hat{a} \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle]. \quad (28)$$

When $\alpha \rightarrow \exp(i\theta)\alpha$, where $0 \leq \theta \leq 2\pi$, we have

$$\begin{aligned} \langle \hat{a}^\dagger \rangle &\rightarrow \exp(-i\theta) \langle \hat{a}^\dagger \rangle, \\ \langle \hat{a}^{\dagger 2} \rangle &\rightarrow \exp(-i2\theta) \langle \hat{a}^{\dagger 2} \rangle, \end{aligned}$$

and

$$\langle \hat{a}^\dagger \hat{a} \rangle \rightarrow \langle \hat{a}^\dagger \hat{a} \rangle.$$

The transformation $\alpha \rightarrow -\alpha$ corresponds to $\theta = \pi$. Substituting in Eqs. 27-28 the transformed expressions for the expectation values of \hat{a} , \hat{a}^\dagger and $\hat{a}^\dagger \hat{a}$ and setting $\theta = \pi$, we see that the uncertainties in x and p for the state $|\alpha\rangle$ are same as those of the state $|- \alpha, k, \rangle$. Thus, the quadrature uncertainties for the states defined on the upper-half of the α -plane yield contain the values for the states defined on the lower-half also. Another interesting transformation is $\alpha \rightarrow i\alpha$, which corresponds to rotation by $\pi/2$. Under this transformation the expression for δx for the state $|\alpha, k\rangle$ goes over to that of Δp for the state $|i\alpha\rangle$. The discussion implies that the knowledge of variance in one of the quadratures for all values of α gives also the magnitude of the fluctuation in the other quadrature. The uncertainty profile for the p -quadrature is same as that of the x , except for a rotation of $\pi/2$ about the axis labeled Δx .

The symmetries exhibited in the uncertainty profiles are not restricted to the states $|\alpha, k\rangle$. They are generic to any state $|S, \alpha\rangle$, characterized by a complex

number α , which has the number state expansion

$$|S, \alpha\rangle = \sum_{n=0}^{\infty} \alpha^n S_n |n\rangle. \quad (29)$$

The coefficients S_n are real.

The states $|\alpha, k\rangle$ are nonclassical as they exhibit squeezing in the quadratures and exhibit sub-Poissonian photon statistics. Hence the Wigner function should become negative somewhere on the complex plane. The Wigner function, obtained using the method given in Ref. [31], is given by

$$W(z) = \frac{2 \exp(-zz^*)}{\pi} N_k^2 \sum_{n,m=0}^{\infty} \frac{\alpha^n \alpha^{*m}}{\sqrt{k^{n+m} \left(\frac{1}{k}\right)_n \left(\frac{1}{k}\right)_m}} \sum_{l=0}^{\min(m,n)} \frac{2^{n+m-2l} z^{n-l} z^{*m-l} (-)^l}{l!(n-l)!(m-l)!} \quad (30)$$

We have plotted the Wigner function for the state $|2.5, .5\rangle$ which exhibits squeezing in x quadrature (see Fig. 2b). As expected, there are valleys of negative values of the Wigner function.

2.2 Group-theoretic coherent states

The group-theoretic coherent states for the algebra of operators defined in Eq.(10) are constructed by the action of the unitary operator $\exp(\alpha A_+ - \alpha^* A_-)$ on the vacuum state $|0\rangle$. Denoting these states as $|\alpha, k\rangle_p$, where the suffix p stands for "Perelomov state", we have

$$|\alpha, k\rangle_p = \exp(\alpha A_+ - \alpha^* A_-) |0\rangle. \quad (31)$$

To get the number state expansion for the rhs of the above equation, we use the following disentangled form, derived using the method described in [32, 33], for the unitary operator,

$$\begin{aligned} \exp(\alpha A_+ - \alpha^* A_-) &= \exp(\beta A_+) \exp(\gamma A_0) \exp(\delta A_-), \\ \beta &= \frac{\exp(i\theta)}{\sqrt{k}} \tanh(\lambda \sqrt{k}), \\ \gamma &= -\frac{2}{k} \log[\cosh(\lambda \sqrt{k})], \\ \delta &= -\beta^*. \end{aligned} \quad (32)$$

In the above expressions, λ and θ are respectively the modulus and argument of α . In the limit of $k \rightarrow 0$, the above expression reduces to the familiar form $\exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = \exp(-\lambda^2/2) \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a})$.

The disentangled form of $\exp(\alpha A_+ - \alpha^* A_-)$, as given Eq. (32), gives the number state expansion

$$|\alpha, k\rangle_p = (1 - k|\beta|^2)^{\frac{1}{2k}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \sqrt{k^n \left(\frac{1}{k}\right)_n} |n\rangle \quad (33)$$

The states are normalizable for all values of α as $k|\beta|^2 = \tanh^2(\sqrt{k}\lambda) \leq 1$ for any α . The disentangled form implies that the states can as well be obtained by the action of $\exp(\beta A_+)$ on the vacuum state $|0\rangle$ and normalizing the resultant state. Of course, this is possible only for the vacuum state as A_- annihilates the vacuum and $\exp(\gamma A_0)$ introduces an overall phase. In the limit of $k \rightarrow 1$, the states become the well known phase states[34].

The inner product of $|\beta', k\rangle_p$ with $|\beta, k\rangle_p$ is

$${}_p\langle\beta, k|\beta'\rangle_p = [(1 - k|\beta|^2)(1 - k|\beta'|^2)]^{\frac{1}{2k}} (1 - k\beta^*\beta')^{-\frac{1}{k}}. \quad (34)$$

The resolution of identity by the states $|\alpha, k\rangle_p$ is written as

$$\frac{1-k}{\pi} \int_{|\beta|^2 \leq \frac{1}{k}} |\alpha, k\rangle_p {}_p\langle\alpha, k| \frac{d^2\beta}{1 - k|\beta|^2} = I. \quad (35)$$

The range of integration is restricted to a disc of radius $\frac{1}{\sqrt{k}}$ in the complex β -plane. If we use the relation $\sqrt{k}|\beta| = \tanh(\sqrt{k}|\alpha|)$, the finite range of integration in the β -plane goes over to integration over the entire α -plane. The resolution of identity enables us to write an arbitrary state $|\psi\rangle$ in terms of $|\alpha, k\rangle_p$ as

$$|\psi\rangle = \frac{1-k}{\pi} \int \frac{d\beta}{1 - k|\beta|^2} g(\alpha^*) |\alpha, k\rangle_p, \quad (36)$$

in which we have used the definition $g(\alpha^*) = {}_p\langle\alpha, k|\psi\rangle$. In the space of $(1 - k|\alpha|^2)^{\frac{1}{2k}} g(\alpha^*)$, the operators of the algebra are

$$A_- = \frac{d}{d\alpha^*}, \quad A_+ = k\alpha^{*2} \frac{d}{d\alpha^*} + \alpha^*, \quad \text{and} \quad A_0 = k\alpha^* \frac{d}{d\alpha^*} + \frac{1}{2}. \quad (37)$$

In the limit of $k \rightarrow 0$, the differential operator realizations of the interpolating algebra in the respective spaces, namely Hilbert space of composed of functions $N_k^{-1}f(\alpha^*)$ and $N_p^{-1}g(\alpha^*)$, yield $\frac{d}{d\alpha^*}$ and α^* and $\frac{1}{2}$. This is to be expected as the algebraic and group-theoretic coherent states are the same in the limit of vanishing k . The representation space contains the entire functions $\exp(|\alpha|^2/2)\langle\alpha|\psi\rangle$.

The group-theoretic coherent states are always super-Poissonian ($Q > 1$). We have shown in Fig. 4 the α -dependence of the Q parameter. Quadrature squeezing has also been studied for these states. As the states $|\alpha, k\rangle_p$ are normalizable only for $\beta \leq 1$, we have studied the squeezing for β lying within the unit circle. The figures 5a-5d give the uncertainty in x as a function of β for $k = .25, .5, .75$ and 1 respectively. As in the case of algebraic coherent states $|\alpha, k\rangle$, the uncertainty profiles exhibit symmetry when $\alpha \rightarrow -\alpha$. Also, the corresponding profiles for p can be obtained by rotating the figures 5a-5d by $\pi/2$ about the (Δx) -axis.

3 Coherent states, phase states and W_3 algebra

In this section we introduce a nonsymmetric set of operators to realize W_3 algebra and construct the relevant coherent states. Consider the operators A_+ , I and $B_- = \frac{1}{\sqrt{1+k\hat{a}^\dagger\hat{a}}}\hat{a}$. These operators satisfy $[B_-, A_+] = I$ for all values of k and provide a realization for the W_3 algebra. The set of operators, however, is not symmetric except when $k = 0$ and in that case we recover the creation and annihilation operators of the harmonic oscillator. The operator B_- is constructed using the method of Shanta *et al*[35]. The algebraic coherent states for this algebra are the eigenstates of B_- . Denoting the eigenstates by $|\alpha, k\rangle\rangle$ and using the definition

$$B_-|\alpha, k\rangle\rangle = \alpha|\alpha, k\rangle\rangle, \quad (38)$$

the number state expansion is

$$|\alpha, k\rangle\rangle = \frac{1}{(1 - k|\alpha|^2)^{\frac{1}{2k}}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{k^n \left(\frac{1}{k}\right)_n} |n\rangle \quad (39)$$

The states are normalizable provided $|\alpha|^2 \leq 1/k$. As $[B_-, A_+] = I$, the unnormalized eigenstates of B_- can be written as

$$|\alpha, k\rangle\rangle = \exp(\alpha A_+)|0\rangle. \quad (40)$$

These states can be identified with $|\alpha, k\rangle_p$ if we set $\alpha = \sqrt{k}\beta$. In the limit of $k \rightarrow 0$, the state $|\alpha, k\rangle\rangle$ becomes the coherent state $|\alpha\rangle$ and when $k \rightarrow 1$ we get the phase states as eigenstates. For other values of k , the states $|\alpha, k\rangle\rangle$ interpolate between the coherent states and the phase states. Thus, the group-theoretic coherent states for the interpolating algebra have been written as the algebraic coherent states of another algebra.

Another set of operators which are closed under commutation consists of A_- , I and $B_+ = B_-^\dagger$. These operators are obtained by taking the adjoint of the operators defined in the beginning of this section. The algebraic coherent states for this algebra are the eigenstates of A_- and they have already been discussed in Section II. The relation $[A_-, B_+] = I$ implies that the unnormalized eigenstates of A_- are obtained by a nonunitary deformation of the vacuum state as follows:

$$|\alpha, k\rangle = \exp(\alpha B_+)|0\rangle. \quad (41)$$

The result shows that the algebraic coherent states for the interpolating algebra can be written as nonunitarily-deformed vacuum state.

4 Summary

An algebra that interpolates $SU(1,1)$ and W_3 algebras has been introduced. The interpolation is made possible by the introduction of the real parameter k in

the elements of the $SU(1,1)$ algebra. The coherent states, both algebraic and group-theoretic, for the general algebra have been constructed and the states are overcomplete. Differential operator representation of the elements of the algebra have been constructed in suitable spaces of entire functions. In the limit of $k \rightarrow 0$, the states become the usual coherent states. Algebraic as well as group-theoretic coherent states exhibit squeezing in the quadratures and hence both are nonclassical. While the former exhibit sub-Poissonian statistics the later are super-Poissonian.

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List of Figures

Fig. 1 Variation of Q parameter as a function of α for the states $|\alpha, k\rangle$.

Fig. 2 Uncertainty in x for the algebraic coherent states for all values of $|\alpha| \leq 2.5$. (a) $k = .25$, (b) $k = .5$, (c) $k = .75$ and (d) $k = 1.0$. Regions of squeezing correspond to those points where the uncertainty falls below 0.5, the coherent state value.

Fig. 3 Plot of the Wigner function $W(z)$ for the state $|2.5, 0.5\rangle$.

Fig. 4 The Q parameter as a function of α for the states $|\alpha, k\rangle_p$. Different curves correspond to different values of k . (a) $k = .25$, (b) $k = .5$, (c) $k = .75$ and (d) $k = 1.0$.

Fig. 5 The x -quadrature fluctuations as a function of β . Squeezing occurs for those values of β where fluctuations are less than 0.5. (a) $k = .25$, (b) $k = .5$, (c) $k = .75$ and (d) $k = 1.0$.









